A real-space renormalization group approach to the ferromagnetic Potts model on the two-dimensional octagonal quasi-periodic tiling

A real-space renormalization group approach to the ferromagnetic Potts model on the two-dimensional octagonal quasi-periodic tiling

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Abstract. A one-step real-space renormalization group (RSRG) transformation is used to study the ferromagnetic (FM) q-state Potts model on the two-dimensional (2D) octagonal quasi-periodic tiling (OQT). The critical exponents of the correlation length v for different values of q and the critical temperature of the Ising model are obtained. The results are shown to be not sensitive to the choice of parameters. The comparison of the results with previous results for the OQT and the square lattice (SQL) seems to show that the universal classes of the q-state Potts models on the OQT and the SQL are the same for the range from $q = 1$ to $q = 3$, in accordance with previous research.

1. Introduction

The Ising model on 2D periodic lattices has received extensive study [1–3], while research on 2D quasi-periodic lattices only began with the discovery of quasi-crystals [4]. Most of the research has been focused on the 2D Penrose tiling lattice (PTL) [5]. Both the Monte Carlo (MC) method [6, 7] and the RSRG method [8–10], as well as a momentum-space RG approach [11], suggested that the Ising model on the PTL belongs to the same universal class as those on the periodic lattices. A MC simulation [12] suggested that the Ising model on the OQT belongs to the same universal class, too. Yet there was also some evidence [13] showing that different quasi-periodicities may lead to different universal classes.

As regards the Potts model [14] on quasi-lattices, there were fewer investigations (see [15] and references therein) reported; thus further studies are still needed.

In this article we have studied the q-state Potts model [14] on the 2D OQT, using a modification of the RSRG transformation introduced by Sire and Bellisard [16] in their study of the electron spectrum on this lattice.

The article is organized as follows. In section 2, we introduce the RSRG transformation. In section 3, the numerical results are obtained and compared with the previous results for the Ising model on the OQT [12] and the exact results for q-state Potts models on the SQL [17]. In section 4 we give our conclusions.

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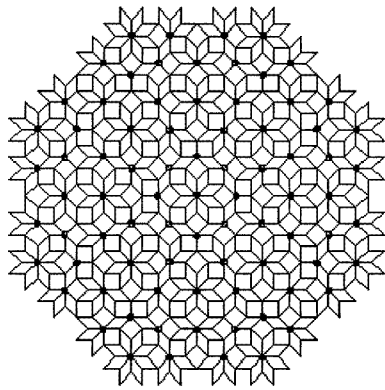
2. The RSRG transformation

The Hamiltonian of the Potts model [14] is

$$H(\sigma) = -\sum_{ij} J_{ij}\delta(\sigma_i, \sigma_j) \tag{1}$$

in which the spin variable σ_i can take q different values, $\delta(\sigma_i, \sigma_j)$ is the Kronecker δ-function, and J_{ij} is the coupling constant for the sites i and j. We study the FM case corresponding to $J_{ij} > 0$.

It is easy to prove [18] that the $q = 1$ case is equivalent to the bond percolation (BP) problem, while the $q = 2$ case is equivalent to the Ising model.

The OQT has eightfold symmetry [16]. It is made up of two tile units [16]: a square, and a rhombus with the sharp angle $45°$ (see figure 1). In this lattice, there are seven types of site (see figure 2).

Figure 1. A part of the OQT showing its self-similarity. The stressed sites are the sites remaining after the self-similar transformation is applied once to the original lattice.
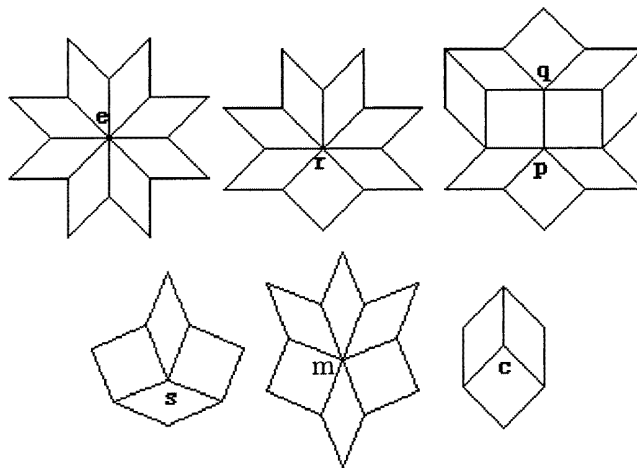

Figure 2. The seven types of site on the OQT.

The OQT is self-similar, and can be transformed into itself in one step [16] with the unit length scale of the lattice inflated by a factor $b = \sqrt{2} + 1$ (see figure 1). In this transformation, three kinds of site, c, s, q, are decimated.

As the incidence of the edges connecting a site c to a site s or q is the largest on the OQT [16], it is natural to choose this kind of bond as one parameter (denoted by t_1) and the bonds along edges of other kinds as another parameter (denoted by t_2), as Sire and Bellisard did in their paper [16].

The inflated tile may have 'effective' bonds along the diagonal lines even when the pre-renormalized tile does not (see figure 1). Physically speaking, the diagonal bonds can be interpreted as 'next-nearest-neighbour' interactions. Therefore, the diagonal bonds of the tiles cannot be neglected in the RG transformation.
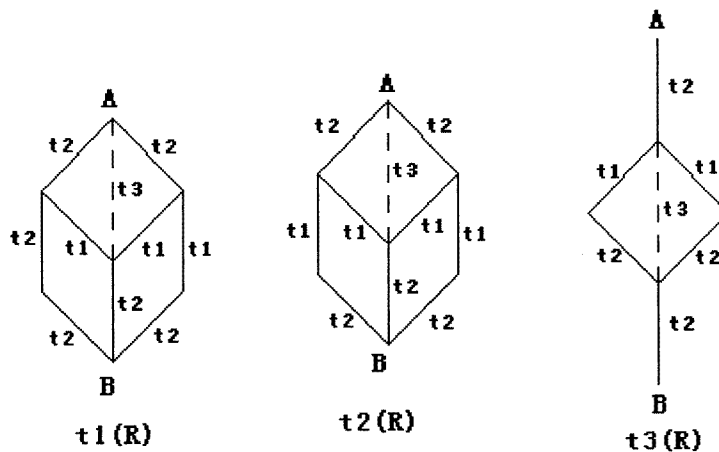
Figure 3. The graphs of renormalized bonds in the three-parameter RG. The labels 't_i(R)' below the graphs denote the renormalized parameters (or bonds) in the inflated lattice to which the graphs above them refer. The labels 't_i' in the graphs represent the parameters in the original lattice, which the bond that the label is attached to denotes. In each graph, the effective TT (see the text) between the two sites A and B equals the TT of the corresponding renormalized bond.

One can see in figure 1 that the two diagonal lines of a square tile are distinguishable: one diagonal line connects at least one site of type c while the other one connects two sites neither of which is of type c. It is apparent in figure 3 that the former directly increases the connectivity between the sites A and B. So, as a first step this diagonal bond is taken as a third parameter, denoted by t_3.

After some analysis, the three renormalized bonds are obtained as shown in figure 3.

As it is well known that the result obtained by the RSRG method may depend strongly on the choice of the parameters, it is necessary to increase the number of parameters step by step to test the stability and reliability of our method.

Firstly, the other diagonal line of the square tile is added (denoted by t_4), leading to the four-parameter RG transformation shown in figure 4.

As a second step, the shorter and longer diagonal bonds of a unit rhombus are introduced as two new parameters, denoted by t_5 and t_6, respectively. The six renormalized bonds are shown in figure 5.

As a further test of our result, another parameter (denoted by t_7) is added. The definition of t_7 and the graph of the renormalized bond t_7(R) are shown in figure 6. This parameter
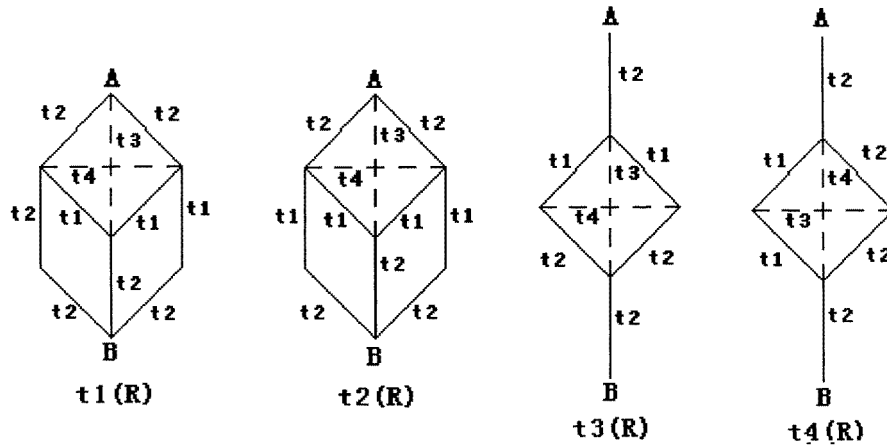
Figure 4. The graphs of the renormalized bonds in the four-parameter RG.
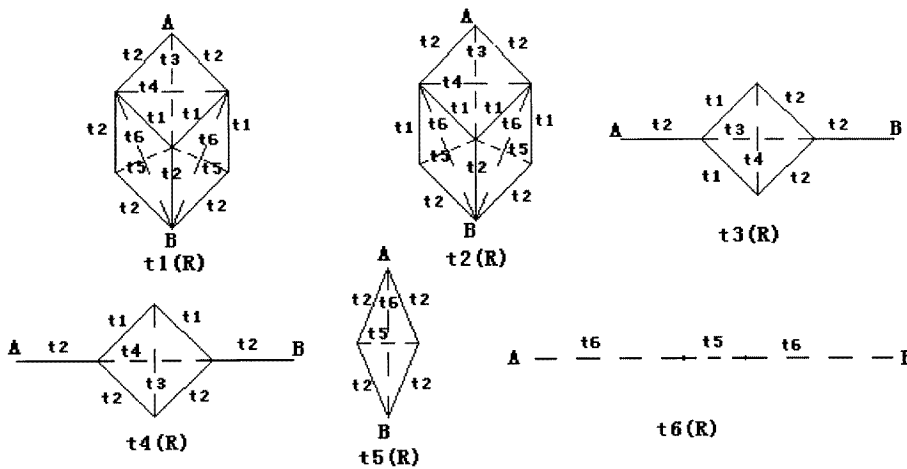

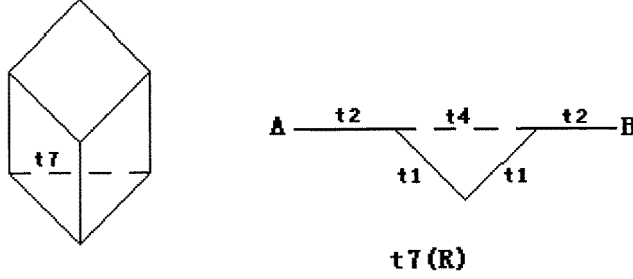**Figure 5.** The graphs of the renormalized bonds in the six-parameter RG.

only influences the renormalization of t_1 and t_2. The graphs of the renormalized bonds $t_1(R)$ and $t_2(R)$ can in this case be obtained just by adding t_7, according to its definition shown in figure 6, into their graphs shown in figure 5. The graphs of the other four renormalized bonds are just the same as in figure 5.

To make the RG transformation, we use the break–collapse method (BCM) [17] which is briefly illustrated below.

In the BCM introduced in reference [17], a convenient parameter called the 'thermal transmissivity (TT)' is defined as follows:

$$t_{ij} = \frac{1 - \exp(-q J_{ij}/k_B T)}{1 + (q-1)\exp(-q J_{ij}/k_B T)} \tag{2}$$

in which k_B is the Boltzmann constant and T is the temperature.

Definition of t7

Figure 6. The definition of the bond t_7 (in the original lattice) introduced in the seven-parameter RG, and the graph of the renormalized one.

Since we are studying the FM case ($J_{ij} > 0$), it is obvious that $t_{ij} = 0$ corresponds to $T = +\infty$ (or $J = +0$) while $t_{ij} = 1$ corresponds to $T = +0$ (or $J = +\infty$).

In the case where $q = 1$, this parameter is equal to the occupation probability of the BP problem [17] while it is the hyperbolic tangent function of the reduced coupling constant $K_{ij} = J_{ij}/k_B T$ in the case where $q = 2$.

In the following RG transformation, the TT is considered instead of the coupling constant, for the two parameters are connected by a one-to-one function which is non-singular except when the coupling constant J_{ij} tends to zero or infinity.

In the BCM [17], the effective TT between two sites A and B connected through a given graph G, denoted by $t_{AB}(G)$, is a fraction with the denominator $D(G)$ and the numerator $N_{AB}(G)$:

$$t_{AB}(G) = \frac{N_{AB}(G)}{D(G)}. \tag{3}$$

For each single bond i connecting two sites A and B, we define $D_i = 1$ and $N_i = t_i$.

For a set of bonds connecting two sites in series, the denominator D_s and numerator N_s of the effective TT are shown to be [17]

$$N_s = \prod_i N_i \tag{4}$$

$$D_s = \prod_i D_i. \tag{5}$$

For bonds connecting two sites in parallel, the denominator N_p and the numerator D_p are [17]

$$N_p = \left\{ \prod_i [D_i + (q-1)N_i] - \prod_i (D_i - N_i) \right\} \Big/ q \tag{6}$$

$$D_p = \left\{ \prod_i [D_i + (q-1)N_i] + (q-1) \prod_i (D_i - N_i) \right\} \Big/ q. \tag{7}$$

For a complex graph G connecting sites A and B which is not simply a combination of parallel and (or) serial bonds, choosing one bond with the denominator D_j and the numerator N_j leads to the following results [17]:

$$N_{AB}(G) = (D_j - N_j)N_{AB}(G_j^b) + N_j N_{AB}(G_j^c) \tag{8}$$

$$D(G) = (D_j - N_j)D(G_j^b) + N_j D(G_j^c) \tag{9}$$

in which G_j^b is the graph obtained from the original graph G by breaking the bond j (or setting $t_j = 0$), and G_j^c is the graph obtained from G by setting the two sites that the bond j connects to be equivalent (or setting $t_j = 1$).

It is clear that with the recursive use of the above method, any given graph can be reduced to a simple graph consisting of only series and (or) parallel bonds. Thus one can obtain the effective TT of any given graph connecting two sites using the BCM method.

It should be noted that with the use of the BCM shown above one can, at least in principle, derive the analytic forms of the RG transformation. However, the derivation is very complicated. Thus this process is usually transformed into a recursive program and left for the computer to finish numerically.

3. Numerical results and discussion

Using standard RG analysis, it is easy to obtain the fixed point, the thermal rescaling factor y_T, and the critical exponent of the correlation length $\nu = 1/y_T$.

Table 1. Comparison of the critical exponents of the Potts models on the OQT and the exact result on the square lattice. ν_{SQ} refers to the exact result for the SQL. ν_{OQT}^A, ν_{OQT}^B, ν_{OQT}^C, and ν_{OQT}^D refer to the exponents of the OQT obtained in the three-, four-, six-, and seven-parameter RG, respectively. ν_{OQT}^{MC} refers to the exponents of the OQT obtained by the MC method.

q	1	2	3	4
ν_{OQT}^A	1.2663	0.9275	0.8062	0.7417
ν_{OQT}^B	1.2672	0.9387	0.8137	0.7469
ν_{OQT}^C	1.2427	0.9716	0.8513	0.7806
ν_{OQT}^D	1.2539	0.9737	0.8520	0.7809
ν_{OQT}^{MC}		0.9940		
ν_{SQ}	$4/3 = 1.3333$	1	$5/6 = 0.8333$	$2/3 = 0.6667$

The critical exponents ν for the $q = 1, 2, 3, 4$ cases are listed in table 1, together with the exact results for the SQL and the MC results for the Ising model (the $q = 2$ Potts model) on the OQT [12]. One can see that the RSRG results are not very sensitive to the number of parameters, and are close to the exact results on the SQL.

The case when $t_1 = t_2$ and all of the other bonds are broken is just the case studied by MC simulation [12]. Our result in this case gives the reduced critical temperatures $k_B T_c/J \approx 2.31, 2.31, 2.37, 2.37$ for the three-, four-, six-, seven-parameter RG, respectively. It seems that the result tends to approach to the MC simulation result [12] $k_B T_c/J \approx 2.39$ as the number of parameters increases.

To make a further test of the stability of the method, the numerical curves for $\nu(q)$ for the range $q = 1$ to $q = 4$ are calculated (the step length of q is taken as 0.05). The results are plotted in figure 7 together with the exact result on the SQL [17]. One can see that the four curves obtained by the RSRG method are close to each other over the whole range (curve A and curve B, and curve C and curve D are nearly indistinguishable), which may serve as evidence of the stability of our results.

Moreover, the four curves are quite close to the exact curve for the SQL in the range from $q = 1$ to $q = 3$, and seem to have a tendency to approach it with increase of the parameter number. From table 2 one can see this systematic tendency clearly. Thus it
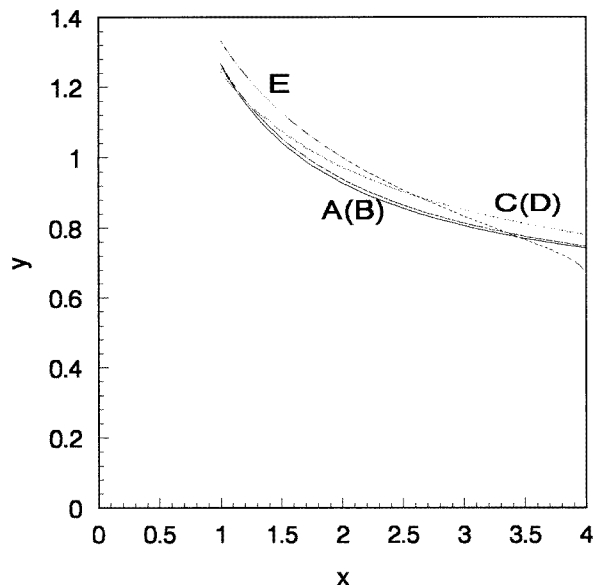
Figure 7. The curves for the critical exponent $v(q)$ on the OQT obtained by the three-parameter (curve A), four-parameter (curve B), six-parameter (curve C), and seven-parameter RG transformations (curve D) and the exact results on the SQL (curve E). In the figure, the x-coordinate is q and the y-coordinate is $v(q)$.

Table 2. A more precise comparison of the critical exponents of the Potts models on the OQT with the exact result on the square lattice. v_{SQ} refers to the exact result for the SQL. v_{OQT}^A, v_{OQT}^B, v_{OQT}^C, and v_{OQT}^D refer to the exponents of the OQT obtained in the three-, four-, six-, and seven-parameter RG, respectively.

q	v_{OQT}^A	v_{OQT}^B	v_{OQT}^C	v_{OQT}^D	v_{SQ}
1.3	1.1129	1.1232	1.1295	1.1356	1.1965
1.4	1.0757	1.0870	1.1000	1.1050	1.1604
1.5	1.0433	1.0551	1.0734	1.0776	1.1278
1.6	1.0147	1.0267	1.0493	1.0493	1.0529
1.7	0.9893	1.0012	1.0273	1.0305	1.0707
1.8	0.9665	0.9783	1.0072	1.0100	1.0454
1.9	0.9460	0.9576	0.9888	0.9911	1.0219
2.0	0.9275	0.9387	0.9716	0.9737	1.0000
2.1	0.9106	0.9214	0.9558	0.9576	0.9794
2.2	0.8951	0.9055	0.9410	0.9426	0.9600
2.3	0.8809	0.8909	0.9273	0.9287	0.9416
2.4	0.8678	0.8774	0.9144	0.9156	0.9242
2.5	0.8556	0.8649	0.9023	0.9034	0.9076
2.6	0.8443	0.8532	0.8909	0.8919	0.8917

seems that our results give some evidence that the q-state Potts models on the OQT and SQL belong to the same universal class over the range from $q = 1$ to $q = 3$.

In the vicinity of $q = 4$, our RSRG results show a systematic deviation from the exact results for the SQL. This may be a common limitation of the RSRG method; the RSRG results for the SQL did not agree with the exact results over this range, either [17].

4. Conclusions

In this paper, the q-state Potts models on the OQT are studied. In order to reduce the influence of the general limitation of the RSRG method, i.e., the dependence on the choice of parameters, four RSRG transformations with different choices of parameters are considered.

As shown in table 1 and figure 7, the critical exponents ν obtained are not sensitive to change of the number of parameters considered.

The reduced critical temperature of the Ising model calculated seems to tend to reach the MC simulation result [12] with increase of the parameter number.

Moreover, with increase of the parameter number the curve in figure 7 seems to approach the exact result on the SQL for the range from $q = 1$ to $q = 3$. This systematic tendency (see table 2) may be considered as support for the reliability of our RG method.

A systematic deviation from the exact results on the SQL with a maximum relative difference of about 17% in the vicinity of $q = 4$ may be attributed to a common limitation of the RSRG method.

Finally, it should be noted that our RSRG method is not a rigorous one, and further studies are still needed to obtain more rigid conclusions.

Acknowledgments

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